

On Matrix Realizations of the Contact Superconformal Algebra $\hat{K}'(4)$ and the Exceptional $N = 6$ Superconformal Algebra

Elena Poletaeva

School of Mathematics, Institute for Advanced Study, Princeton and
Department of Mathematics, University of Texas - Pan American (permanent address),
Email: elena@math.ias.edu and elenap@utpa.edu

Abstract. The superalgebra $\hat{K}'(4)$ and the exceptional $N = 6$ superconformal algebra have “small” irreducible representations in the superspaces $V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $N = 2$ and 3 , respectively. For $\mu \in \mathbb{C} \setminus \mathbb{Z}$ they are associated to the embeddings of these superalgebras into the Lie superalgebras of pseudodifferential symbols on the supercircle $S^{1|N}$. In this work we describe $\hat{K}'(4)$ and the exceptional $N = 6$ superconformal algebra in terms of matrices over a Weyl algebra. Correspondingly, we obtain realizations of their representations in V^μ for $\mu = 0$.

Keywords. Superconformal algebra, pseudodifferential symbols, Poisson superalgebra, Weyl algebra.

AMS (MOS) subject classification: 17B68, 17B65, 81R10

1 Introduction

This work is a continuation of [20, 21].

Recall that a *superconformal algebra* is a simple complex Lie superalgebra spanned by the coefficients of a finite family of pairwise local fields

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

one of which is the Virasoro field $L(z)$ [3, 9–11]. Superconformal algebras play an important rôle in the string theory and conformal field theory. They can also be described in terms of derivations of the associative superalgebra $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in N variables. The Lie superalgebra $K(N)$ of contact vector fields with Laurent polynomials as coefficients is spanned by 2^N fields [3, 6, 7, 10]. It is also known as the $SO(N)$ superconformal algebra [1]. $K(N)$ is simple if $N \neq 4$, if $N = 4$, then the derived Lie superalgebra $K'(4)$ is simple. The nontrivial central extensions of $K(1)$, $K(2)$, and $K'(4)$ are well-known: they are isomorphic to the Neveu-Schwarz superalgebra, the “ $N = 2$ ” superconformal algebra, and the “big $N = 4$ ” superconformal algebra [1]. $K(6)$ contains the exceptional $N = 6$ superconformal algebra, also denoted by CK_6 , as a sub-superalgebra. Note that CK_6 is “one half” of $K(6)$: it is spanned by 32 fields [3, 4, 6, 12, 15, 22–24].

In [16, 17] Martinez and Zelmanov obtained CK_6 as a particular case of their construction of superalgebras $CK(R, d)$, where R is an associative commutative superalgebra with an even derivation d .

Our approach is based on the realization of $K(2N)$ in terms of pseudodifferential symbols on the circle extended by N odd variables. It is well-known that a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra [2]. Analogously, $K(2N)$ can be embedded into the Poisson superalgebra $P(2N)$ of pseudodifferential symbols on the supercircle $S^{1|N}$ [20, 21]. There exists a family $P_h(2N)$ of Lie superalgebras of pseudodifferential symbols on $S^{1|N}$, which contracts to $P(2N)$. There is no embedding of $K(2N)$ into $P_h(2N)$ if $N \geq 3$. It is remarkable that a nontrivial central extension $\hat{K}'(4)$ of $K'(4)$ and CK_6 can be embedded into $P_h(2N)$, where $N = 2$ and 3 , respectively [20, 21].

Associated to these embeddings, there are “small” irreducible representations of $\hat{K}'(4)$ and CK_6 in the superspaces $V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $(\partial/\partial t)^{-1}$ acts as an antiderivative. This requires that $\mu \in \mathbb{C} \setminus \mathbb{Z}$. Nevertheless, the representations of $\hat{K}'(4)$ and CK_6 in V^μ can be defined if $\mu = 0$. In this work we describe these superalgebras in terms of matrices over the Weyl algebra $W = \sum_{i \geq 0} \mathcal{A} d^i$, where $\mathcal{A} = \mathbb{C}[t, t^{-1}]$ and $d = t \partial/\partial t$ (Theorems 1 and 2). This gives realizations of the representations in V^μ for $\mu = 0$.

2 Contact and Poisson superalgebras

A *superconformal algebra* is a complex Lie superalgebra \mathfrak{g} such that

- (1) \mathfrak{g} is simple,
- (2) \mathfrak{g} contains the centerless Virasoro algebra $\text{der } \mathbb{C}[t, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n$ with the commutation relations

$$[L_n, L_m] = (m - n) L_{n+m} \quad (1)$$

as a subalgebra,

- (3) $ad L_0$ is diagonalizable with finite-dimensional eigenspaces:

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i, \quad \mathfrak{g}_i = \{x \in \mathfrak{g} \mid [L_0, x] = ix\}, \quad (2)$$

so that $\dim \mathfrak{g}_i < C$, where C is a constant independent of i [7].

Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N$, and let

$$\Lambda(1, 2N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(2N)$$

be the associative superalgebra with natural multiplication and with the following parity of generators:

$$p(t) = \bar{0}, \quad p(\xi_i) = p(\eta_i) = \bar{1} \text{ for } i = 1, \dots, N.$$

Let $W(2N)$ be the Lie superalgebra of all derivations of $\Lambda(1, 2N)$. By definition,

$$K(2N) = \{D \in W(2N) | D\Omega = f\Omega \text{ for some } f \in \Lambda(1, 2N)\}$$

where $\Omega = dt + \sum_{i=1}^N (\xi_i d\eta_i + \eta_i d\xi_i)$ is a differential *contact* 1-form [3–7, 10, 22–24]. There is a one-to-one correspondence between the differential operators $D \in K(2N)$ and the functions $f \in \Lambda(1, 2N)$. Let $\partial_t, \partial_{\xi_i}$ and ∂_{η_i} stand for $\partial/\partial t, \partial/\partial \xi_i$ and $\partial/\partial \eta_i$, respectively. The correspondence $f \leftrightarrow D_f$ is given by

$$D_f = \Delta(f)\partial_t + (\partial_t f)E - H_f,$$

where

$$E = \sum_{i=1}^N (\xi_i \partial_{\xi_i} + \eta_i \partial_{\eta_i}), \quad \Delta = 2 - E,$$

$$H_f = (-1)^{p(f)+1} \sum_{i=1}^N (\partial_{\xi_i} f \partial_{\eta_i} + \partial_{\eta_i} f \partial_{\xi_i}).$$

The *Poisson algebra* P of *pseudodifferential symbols on the circle* is formed by the formal series

$$A(t, \tau) = \sum_{i=-\infty}^n a_i(t) \tau^i,$$

where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the even variable τ corresponds to ∂_t . The Poisson bracket is defined as follows:

$$\{A(t, \tau), B(t, \tau)\} = \partial_\tau A(t, \tau) \partial_t B(t, \tau) - \partial_t A(t, \tau) \partial_\tau B(t, \tau).$$

An associative algebra P_h , where $h \in (0, 1]$ is a deformation of P . The multiplication in P_h is given as follows:

$$A(t, \tau) \circ_h B(t, \tau) = \sum_{n \geq 0} \frac{h^n}{n!} \partial_\tau^n A(t, \tau) \partial_t^n B(t, \tau).$$

The Lie algebra structure on the vector space P_h is given by

$$[A, B]_h = A \circ_h B - B \circ_h A,$$

so that

$$\lim_{h \rightarrow 0} \frac{1}{h} [A, B]_h = \{A, B\}, \quad (3)$$

see [13, 14, 18, 19]. The *Poisson superalgebra of pseudodifferential symbols on $S^{1|N}$* is $P(2N) = P \otimes \Lambda(2N)$. The Poisson bracket is defined as follows:

$$\begin{aligned} \{A, B\} &= \partial_\tau A \partial_t B - \partial_t A \partial_\tau B + \\ &+ (-1)^{p(A)+1} \sum_{i=1}^N (\partial_{\xi_i} A \partial_{\eta_i} B + \partial_{\eta_i} A \partial_{\xi_i} B). \end{aligned}$$

Let $\Lambda_h(2N)$ be an associative superalgebra with generators $\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N$ and relations

$$\xi_i \xi_j = -\xi_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \xi_j = h \delta_{i,j} - \xi_j \eta_i.$$

Let $P_h(2N) = P_h \otimes \Lambda_h(2N)$ be a superalgebra with the product given by

$$(A_1 \otimes X)(B_1 \otimes Y) = (A_1 \circ_h B_1) \otimes (XY),$$

where $A_1, B_1 \in P_h$ and $X, Y \in \Lambda_h(2N)$. The Lie bracket of $A = A_1 \otimes X$ and $B = B_1 \otimes Y$ is

$$[A, B]_h = AB - (-1)^{p(A)p(B)} BA,$$

and (3) is satisfied. $P_h(2N)$ is the *Lie superalgebra of pseudodifferential symbols on $S^{1|N}$* . There exist embeddings of $\hat{K}'(4)$ and CK_6 into $P_h(2N)$, where $N = 2$ and $N = 3$, respectively [20, 21].

3 Case $\hat{K}'(4)$

The derived superalgebra

$$K'(4) = [K(4), K(4)]$$

is a simple ideal in $K(4)$ of codimension one, defined from the exact sequence

$$0 \rightarrow K'(4) \rightarrow K(4) \rightarrow \mathbb{C} D_{t^{-1} \xi_1 \xi_2 \eta_1 \eta_2} \rightarrow 0.$$

The superalgebra $K'(4)$ is spanned inside $P(4)$ by the 12 fields:

$$L_n = t^{n+1} \tau, \quad X_n^j = t^{n+1} \tau \xi_j, \quad Y_n = t^{n+1} \tau \xi_1 \xi_2, \quad (4)$$

$$L_n^i = t^n \eta_i, \quad X_n^{ji} = t^n \xi_j \eta_i, \quad Y_n^i = t^n \xi_1 \xi_2 \eta_i, \quad (5)$$

where $i, j = 1, 2$, and 4 fields

$$\begin{aligned} F_n^0 &= t^{n-1} \tau^{-1} \eta_1 \eta_2, \\ F_n^i &= t^{n-1} \tau^{-1} \xi_i \eta_1 \eta_2, \quad i = 1, 2, \\ F_n^3 &= t^{n-1} \tau^{-1} \xi_1 \xi_2 \eta_1 \eta_2, \quad n \neq 0. \end{aligned}$$

Note that L_n is a Virasoro field [20, 21]. Let $\hat{K}'(4)$ be one of three independent central extensions of $K'(4)$, such that the corresponding 2-cocycle is

$$\begin{aligned} c(L_n, F_k^3) &= \delta_{n+k,0}, \quad n \neq 0, \\ c(X_n^i, F_k^j) &= (-1)^j \delta_{n+k,0}, \quad 1 \leq i \neq j \leq 2, \\ c(Y_n, F_k^0) &= \delta_{n+k,0}. \end{aligned}$$

The superalgebra $\hat{K}'(4) \subset P_h(4)$ is spanned by the 12 fields (4)–(5) and 4 fields:

$$F_{n,h}^0 = \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2, \quad (6)$$

$$F_{n,h}^i = \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2 \xi_i, \quad i = 1, 2, \quad (7)$$

$$F_{n,h}^3 = \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2 \xi_1 \xi_2 + \frac{h}{n} t^n, \quad n \neq 0, \quad (8)$$

and the central element $h \in P_h(4)$, so that

$$\lim_{h \rightarrow 0} \hat{K}'(4) = K'(4) \subset P(4).$$

Let $V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2)$, where $\mu \in \mathbb{C} \setminus \mathbb{Z}$. We fix $h = 1$, and define a representation of $\hat{K}'(4)$ in V^μ accordingly to the formulas (4)–(8). Namely, ξ_i is the operator of multiplication in $\Lambda(\xi_1, \xi_2)$, η_i is identified with ∂_{ξ_i} , τ^{-1} is identified with an antiderivative, and the central element $1 \in P_{h=1}(4)$ acts by the identity operator. Consider the following basis in V^μ :

$$\begin{aligned} v_m^0(\mu) &= \frac{1}{m+\mu} t^{m+\mu}, & v_m^1(\mu) &= t^{m+\mu} \xi_1, \\ v_m^2(\mu) &= t^{m+\mu} \xi_2, & v_m^3(\mu) &= t^{m+\mu} \xi_1 \xi_2, \quad m \in \mathbb{Z}. \end{aligned}$$

Explicitly, the action of $\hat{K}'(4)$ on V^μ is given as follows:

$$\begin{aligned} L_n(v_m^0(\mu)) &= (n+m+\mu)v_{m+n}^0(\mu), \\ L_n(v_m^i(\mu)) &= (m+\mu)v_{m+n}^i(\mu), \quad i = 1, 2, 3, \\ X_n^i(v_m^0(\mu)) &= v_{m+n}^i(\mu), \quad i = 1, 2, \\ X_n^1(v_m^2(\mu)) &= (m+\mu)v_{m+n}^3(\mu), \\ X_n^2(v_m^1(\mu)) &= -(m+\mu)v_{m+n}^3(\mu), \\ Y_n(v_m^0(\mu)) &= v_{m+n}^3(\mu), \\ L_n^i(v_m^i(\mu)) &= (n+m+\mu)v_{m+n}^0(\mu), \quad i = 1, 2, \\ L_n^1(v_m^3(\mu)) &= v_{m+n}^2(\mu), \quad L_n^2(v_m^3(\mu)) = -v_{m+n}^1(\mu), \\ X_n^{ji}(v_m^i(\mu)) &= v_{m+n}^j(\mu), \quad i, j = 1, 2, \\ X_n^{ii}(v_m^3(\mu)) &= v_{m+n}^3(\mu), \quad i, j = 1, 2, \\ Y_n(v_m^i(\mu)) &= v_{m+n}^3(\mu), \quad i = 1, 2, \\ F_{n,1}^0(v_m^3(\mu)) &= -v_{m+n}^0(\mu), \\ F_{n,1}^1(v_m^2(\mu)) &= -v_{m+n}^0(\mu), \quad F_{n,1}^2(v_m^1(\mu)) = v_{m+n}^0(\mu), \\ F_{n,1}^3(v_m^i(\mu)) &= \frac{1}{n} v_{m+n}^i(\mu), \quad n \neq 0, \quad i = 0, 1, 2, 3. \end{aligned}$$

Naturally, $V^\mu = \oplus_m V_m^\mu$, where $V_m^\mu = t^{m+\mu} \otimes \Lambda(\xi_1, \xi_2)$. A \mathbb{Z} -grading on $\hat{K}'(4)$ is defined by the element $L_0 = t\tau$ of the Virasoro algebra according to (2). We have that

$$\mathfrak{g}_i(V_m^\mu) \subset V_{m+i}^\mu, \quad (9)$$

and $\mathfrak{g}_0 \cong \hat{\mathfrak{sl}}(2|2)$, where the central element is L_0 . Note that $\hat{\mathfrak{sl}}(2|2)$ has the following one-parameter family spin_λ of $(2|2)$ -dimensional irreducible representations:

$$\text{spin}_\lambda : \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \oplus \mathbb{C}L_0 \rightarrow \left(\begin{array}{cc} A & B + \lambda \tilde{C} \\ C & D \end{array} \right) \oplus \mathbb{C}\lambda \cdot 1_{2|2},$$

where $A, B, C, D \in \mathfrak{gl}(2, \mathbb{C})$, $\text{tr}A = \text{tr}D$, $\lambda \in \mathbb{C}$. Let E_{ij} be an elementary 2×2 -matrix. \tilde{C} is determined by the following conditions:

$$\begin{aligned} \text{if } C = E_{ii}, \text{ then } \tilde{C} &= E_{jj}, \text{ where } i \neq j, \\ \text{if } C = E_{ij}, \text{ } i \neq j, \text{ then } \tilde{C} &= -E_{ij}. \end{aligned} \quad (10)$$

According to (9), there is a representation of \mathfrak{g}_0 in V_m^μ for each $m \in \mathbb{Z}$, and $V_m^\mu \cong \text{spin}_{\lambda=m+\mu}$ as \mathfrak{g}_0 -modules.

Note that if $\mu = 0$, we cannot formally define a representation of $\hat{K}'(4)$ in V^μ . Nevertheless, all the formulas for the action of $\hat{K}'(4)$ on vectors $v_m^i(\mu)$, where $i = 0, 1, 2, 3$ and $m \in \mathbb{Z}$, remain true to $\mu = 0$. Thus a representation of $\hat{K}'(4)$ in the superspace

$$V = \text{Span}(v_m^i(0) \mid i = 0, 1, 2, 3 \text{ and } m \in \mathbb{Z}) \quad (11)$$

is well-defined. To obtain a realization of this representation, at first we will describe $\hat{K}'(4)$ in terms of matrices over a Weyl algebra. By definition, a Weyl algebra is

$$W = \sum_{i \geq 0} \mathcal{A} d^i, \quad (12)$$

where \mathcal{A} is an associative commutative algebra and $d : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation of \mathcal{A} with the relations

$$da = d(a) + ad, \quad a \in \mathcal{A}.$$

Set

$$\mathcal{A} = \mathbb{C}[t, t^{-1}], \quad d = L_0 = t\tau. \quad (13)$$

Replacing λ by d in the formulas for spin_λ , we obtain the following theorem.

Theorem 1. Let $\hat{K}'(4) = \oplus_i \mathfrak{g}_i$, where the \mathbb{Z} -grading is given by $L_0 = t\tau$. Then

1) $\mathfrak{g}_0 \cong \hat{\mathfrak{sl}}(2|2)$ is realized as a Lie superalgebra of 4×4 matrices over $\mathbb{C}[d]$ of the type

$$\left(\begin{array}{cc} A & B + d\tilde{C} \\ C & D \end{array} \right) \oplus \mathbb{C}d \cdot 1_{2|2},$$

where A, B, C , and D are 2×2 matrices over \mathbb{C} , $\text{tr}A = \text{tr}D$ and \tilde{C} is determined by the conditions (10). The central element in $\hat{\mathfrak{sl}}(2|2)$ is $L_0 = d \cdot 1_{2|2}$, and the central element in $\hat{K}'(4)$ is $1_{2|2}$.

2) $\hat{K}'(4)$ is a subsuperalgebra of 4×4 matrices over W generated by $\hat{\mathfrak{sl}}(2|2)$ and by all matrices

$$\left(\begin{array}{cc} E_{ij}(a) & 0 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 0 \\ 0 & E_{ij}(a) \end{array} \right),$$

where $a \in \mathcal{A}$ and $1 \leq i \neq j \leq 2$.

3) The standard representation of $\hat{K}'(4)$, realized as matrices over W , in $(2|2)$ -dimensional vector superspace over \mathcal{A} is isomorphic to the above-mentioned representation in the superspace V in the case when $\mu = 0$, see (11).

4 Case CK_6

The exceptional superconformal algebra CK_6 is spanned by the following 32 fields inside $K(6) \subset P(6)$:

$$\begin{aligned} L_n &= t^{n+1}\tau, & G_n^i &= t^{n+1}\tau\xi_i, \quad i = 1, 2, 3, \\ \tilde{G}_n^i &= t^n\eta_i - nt^{n-1}\tau^{-1}\xi_j\eta_i\eta_j, & i &= 1, 2, 3, \\ T_n^{ij} &= t^n\xi_i\eta_j - nt^{n-1}\tau^{-1}\xi_k\xi_i\eta_k\eta_j, & i &\neq j \neq k, \\ T_n^i &= -t^n(\xi_j\eta_j + \xi_k\eta_k) + nt^{n-1}\tau^{-1}\xi_j\xi_k\eta_j\eta_k, & i &= 1, 2, 3, \\ S_n^i &= -t^n\xi_i(\xi_j\eta_j + \xi_k\eta_k) + nt^{n-1}\tau^{-1}\xi_i\xi_j\xi_k\eta_j\eta_k, & i &= 1, 2, 3, \\ \tilde{S}_n^i &= t^{n-1}\tau^{-1}(\xi_j\eta_j - \xi_k\eta_k)\eta_i, & i &= 1, 2, 3, \end{aligned}$$

$$I_n^i = t^{n-1} \tau^{-1} \xi_i \eta_j \eta_k, \quad i = 1, 2, 3, \quad I_n = t^{n+1} \tau \xi_1 \xi_2 \xi_3, \\ J_n^{ij} = t^{n+1} \tau \xi_i \xi_j, \quad \tilde{J}_n^{ij} = t^{n-1} \tau^{-1} \eta_i \eta_j, \quad i < j,$$

where $n \in \mathbb{Z}$, and (i, j, k) is the cycle $(1, 2, 3)$ in the formulas for \tilde{G}_n^i , T_n^i , S_n^i , \tilde{S}_n^i , and I_n^i , see [21]. Note that L_n is a Virasoro field.

CK_6 is spanned inside $P_h(6)$ by the 8 fields: L_n , G_n^i , I_n , and J_n^{ij} , and the following 24 fields:

$$\begin{aligned} \tilde{G}_{n,h}^i &= t^n \eta_i - n \tau^{-1} \circ_h t^{n-1} \eta_i \eta_j \xi_j, \quad i = 1, 2, 3, \\ T_{n,h}^{ij} &= t^n \xi_i \eta_j - n \tau^{-1} \circ_h t^{n-1} \eta_k \eta_j \xi_k \xi_i, \quad i \neq j \neq k, \\ T_{n,h}^i &= -t^n (\xi_j \eta_j + \xi_k \eta_k) + \\ &\quad n \tau^{-1} \circ_h t^{n-1} \eta_j \eta_k \xi_j \xi_k + h t^n, \\ S_{n,h}^i &= -t^n \xi_i (\xi_j \eta_j + \xi_k \eta_k) + \\ &\quad n \tau^{-1} \circ_h t^{n-1} \eta_j \eta_k \xi_i \xi_j \xi_k + h t^n \xi_i, \\ \tilde{S}_{n,h}^i &= \tau^{-1} \circ_h t^{n-1} (\eta_j \eta_i \xi_j - \eta_k \eta_i \xi_k), \quad i = 1, 2, 3, \\ I_{n,h}^i &= \tau^{-1} \circ_h t^{n-1} \eta_j \eta_k \xi_i, \quad i = 1, 2, 3, \\ \tilde{J}_{n,h}^{ij} &= \tau^{-1} \circ_h t^{n-1} \eta_i \eta_j, \quad i < j, \end{aligned} \quad (14)$$

where $n \in \mathbb{Z}$, and (i, j, k) is the cycle $(1, 2, 3)$. We have that $\lim_{h \rightarrow 0} CK_6 = CK_6 \subset P(6)$, see [21].

Let $V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Pi(\Lambda(\xi_1, \xi_2, \xi_3))$, where $\mu \in \mathbb{C} \setminus \mathbb{Z}$, and Π denotes the change of parity. We fix $h = 1$, and define a representation of CK_6 in V^μ according to the formulas (14). Consider the following basis in V^μ :

$$\begin{aligned} v_m^i(\mu) &= \frac{t^{m+\mu}}{m+\mu} \Pi(\xi_i), \quad \hat{v}_m^i(\mu) = t^{m+\mu} \Pi(\xi_j \xi_k), \quad 1 \leq i \leq 3, \\ v_m^4(\mu) &= \frac{t^{m+\mu}}{m+\mu} \Pi(1), \quad \hat{v}_m^4(\mu) = -t^{m+\mu} \Pi(\xi_1 \xi_2 \xi_3), \end{aligned}$$

where $m \in \mathbb{Z}$ and (i, j, k) is the cycle $(1, 2, 3)$ in the formulas for $\hat{v}_m^i(\mu)$. Explicitly, the action of CK_6 on V^μ is given as follows:

$$\begin{aligned} L_n(v_m^i(\mu)) &= (m+n+\mu) v_{m+n}^i(\mu), \\ L_n(\hat{v}_m^i(\mu)) &= (m+\mu) \hat{v}_{m+n}^i(\mu), \\ G_n^i(v_m^4(\mu)) &= (m+n+\mu) v_{m+n}^i(\mu), \\ G_n^i(\hat{v}_m^i(\mu)) &= -(m+\mu) \hat{v}_{m+n}^i(\mu), \\ G_n^i(v_m^j(\mu)) &= \hat{v}_{m+n}^k(\mu), \quad G_n^i(v_m^k(\mu)) = -\hat{v}_{m+n}^j(\mu), \\ \tilde{G}_{n,1}^i(v_m^i(\mu)) &= v_{m+n}^4(\mu), \quad \tilde{G}_{n,1}^i(\hat{v}_m^4(\mu)) = -\hat{v}_{m+n}^i(\mu), \\ \tilde{G}_{n,1}^i(v_m^j(\mu)) &= -(m+\mu) v_{m+n}^k(\mu), \\ \tilde{G}_{n,1}^i(\hat{v}_m^k(\mu)) &= (m+n+\mu) v_{m+n}^j(\mu), \\ T_{n,1}^{ij}(v_m^j(\mu)) &= v_{m+n}^i(\mu), \quad T_{n,1}^{ij}(\hat{v}_m^i(\mu)) = -\hat{v}_{m+n}^j(\mu), \\ T_{n,1}^i(v_m^i(\mu)) &= v_{m+n}^i(\mu), \quad T_{n,1}^i(v_m^4(\mu)) = v_{m+n}^4(\mu), \\ T_{n,1}^i(\hat{v}_m^i(\mu)) &= -\hat{v}_{m+n}^i(\mu), \quad T_{n,1}^i(\hat{v}_m^4(\mu)) = -\hat{v}_{m+n}^4(\mu), \\ S_{n,1}^i(v_m^4(\mu)) &= v_{m+n}^i(\mu), \quad S_{n,1}^i(\hat{v}_m^i(\mu)) = \hat{v}_{m+n}^4(\mu), \\ \tilde{S}_{n,1}^i(\hat{v}_m^j(\mu)) &= v_{m+n}^k(\mu), \quad \tilde{S}_{n,1}^i(\hat{v}_m^k(\mu)) = v_{m+n}^j(\mu), \\ I_{n,1}^i(\hat{v}_m^i(\mu)) &= -v_{m+n}^i(\mu), \quad I_n(v_m^4(\mu)) = -\hat{v}_{m+n}^4(\mu), \\ J_n^{ij}(v_m^4(\mu)) &= \hat{v}_{m+n}^k(\mu), \quad J_n^{ij}(v_m^k(\mu)) = -\hat{v}_{m+n}^i(\mu), \\ \tilde{J}_{n,1}^{ij}(\hat{v}_m^i(\mu)) &= -v_{m+n}^4(\mu), \quad \tilde{J}_{n,1}^{ij}(\hat{v}_m^4(\mu)) = v_{m+n}^k(\mu), \end{aligned}$$

where (i, j, k) is the cycle $(1, 2, 3)$, see [21].

We have that $V^\mu = \oplus_m V_m^\mu$, where $V_m^\mu = t^{m+\mu} \otimes \Pi(\Lambda(\xi_1, \xi_2, \xi_3))$. A \mathbb{Z} -grading in CK_6 is defined by the element $L_0 = t\tau$ of the Virasoro algebra according to (2), so that (9) holds. Note that $\mathfrak{g}_0 \cong \hat{\mathcal{P}}(4)$, where the central element is L_0 and $\mathcal{P}(4)$ is a simple Lie superalgebra defined as follows. Let $\tilde{\mathcal{P}}(4)$ be the Lie superalgebra, which preserves the odd nondegenerate supersymmetric bilinear form $\text{antidiag}(1_4, 1_4)$ on the $(4|4)$ -dimensional complex superspace. Thus

$$\tilde{\mathcal{P}}(4) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A \in \mathfrak{gl}(4, \mathbb{C}), B^t = B, C^t = -C \right\}.$$

$\mathcal{P}(4)$ is a subsuperalgebra of $\tilde{\mathcal{P}}(4)$ such that $A \in \mathfrak{sl}(4, \mathbb{C})$, see [8]. $\hat{\mathcal{P}}(4)$ is a nontrivial central extension of $\mathcal{P}(4)$. It is known that $\hat{\mathcal{P}}(4)$ has a family spin_λ of $(4|4)$ -dimensional irreducible representations:

$$\text{spin}_\lambda : \left(\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \oplus \mathbb{C} L_0 \right) \rightarrow \left(\begin{pmatrix} A & B - \lambda \tilde{C} \\ C & -A^t \end{pmatrix} \oplus \mathbb{C} \lambda \cdot 1_{4|4} \right),$$

where $\lambda \in \mathbb{C}$, $1_{4|4}$ is the identity matrix, and \tilde{C} is determined by the following condition:

$$\begin{aligned} \text{if } C_{ij} &= E_{ij} - E_{ji}, \text{ then } \tilde{C}_{ij} = C_{kl}, \text{ so that} \\ \text{the permutation } (1, 2, 3, 4) &\mapsto (i, j, k, l) \text{ is even,} \end{aligned} \quad (15)$$

cf. [6] and [22–24]. According to (9), there is a representation of \mathfrak{g}_0 in V_m^μ for each $m \in \mathbb{Z}$, and $V_m^\mu \cong \text{spin}_{\lambda=m+\mu}$ as \mathfrak{g}_0 -modules.

Similarly to the case of $\hat{K}'(4)$, all the formulas for the action of CK_6 on vectors $v_m^i(\mu)$, $\hat{v}_m^i(\mu)$, where $1 \leq i \leq 4$ and $m \in \mathbb{Z}$, remain true to $\mu = 0$. Thus a representation of CK_6 in the superspace

$$V = \text{Span}(v_m^i(0), \hat{v}_m^i(0) \mid 1 \leq i \leq 4 \text{ and } m \in \mathbb{Z}) \quad (16)$$

is well-defined. To obtain a realization of this representation, we will use the Weyl algebra W defined in (12) and (13). Replacing λ by d in the formulas for spin_λ , we obtain the following theorem, cf. [17].

Theorem 2. Let $CK_6 = \oplus_i \mathfrak{g}_i$, where the \mathbb{Z} -grading is given by $L_0 = t\tau$. Then

1) $\mathfrak{g}_0 \cong \hat{\mathcal{P}}(4)$ is realized as a Lie superalgebra of 8×8 matrices over $\mathbb{C}[d]$ of the type

$$\left(\begin{pmatrix} A & B - d\tilde{C} \\ C & -A^t \end{pmatrix} \oplus \mathbb{C} d \cdot 1_{4|4} \right),$$

where A , B , and C are 4×4 matrices over \mathbb{C} , $\text{tr} A = 0$, $B^t = B$, $C^t = -C$, and \tilde{C} is determined by the condition (15). The central element in $\hat{\mathcal{P}}(4)$ is $L_0 = d \cdot 1_{4|4}$.

2) CK_6 is a subsuperalgebra of 8×8 matrices over W generated by $\hat{\mathcal{P}}(4)$ and by all matrices

$$\left(\begin{pmatrix} E_{ij}(a) & 0 \\ 0 & -E_{ji}(a) \end{pmatrix} \right), \text{ where } a \in \mathcal{A} \text{ and } 1 \leq i \neq j \leq 4.$$

3) The standard representation of CK_6 , realized as matrices over W , in $(4|4)$ -dimensional vector superspace over \mathcal{A} is isomorphic to the above-mentioned representation in the superspace V in the case when $\mu = 0$, see (16).

5 Acknowledgements

This material is based upon work supported by the National Science Foundation under agreement *No. DMS – 0111298*. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

The author is grateful to the Institute for Advanced Study for the hospitality and support during term II of the academic year 2006–2007. She wishes to thank the organizers of the 5th International Conference on Differential Equations and Dynamical Systems. She is also grateful to V. Serganova for very useful discussions.

6 References

- [1] M. Ademollo, L. Brink, A. D’Adda et al., Dual strings with $U(1)$ colour symmetry, *Nucl. Phys.*, B **111**, (1976) 77–110.
- [2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York, 1989.
- [3] S.-J. Cheng and V. G. Kac, A new $N = 6$ superconformal algebra, *Commun. Math. Phys.*, **186**, (1997) 219–231.
- [4] S.-J. Cheng and V. G. Kac, Structure of some \mathbb{Z} -graded Lie superalgebras of vector fields, *Transform. Groups*, **4**, (1999) 219–272.
- [5] B. Feigin and D. Leites, New Lie superalgebras of string theories, in *Group-Theoretical Methods in Physics*, edited by M. Markov et al., Nauka, Moscow **1**, (1983) 269–273. (English translation Gordon and Breach, New York, 1984).
- [6] P. Grozman, D. Leites, and I. Shchepochkina, Lie superalgebras of string theories, *Acta Math. Vietnam.*, **26**, (2001) 27–63; hep-th/9702120.
- [7] V. G. Kac and J. W. van de Leur, On classification of superconformal algebras, in *Strings-88*, edited by S. J. Gates et al., world Scientific, Singapore, (1989) 77–106.
- [8] V. G. Kac, Lie superalgebras, *Adv. Math.*, **26**, (1977) 8–96.
- [9] V. G. Kac, Classification of supersymmetries, *Proceedings of the International Congress of Mathematicians*, **1**, Beijing (2002) 319–344, Higher Ed. Press, Beijing, 2002.
- [10] V. G. Kac, Superconformal algebras and transitive group actions on quadrics, *Commun. Math. Phys.*, **186**, (1997) 233–252. Erratum: **217**, (2001) 697–698.
- [11] V. G. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10, AMS, Providence, RI, 1996. Second edition, 1998.
- [12] V. G. Kac, Classification of infinite-dimensional simple linearly compact Lie superalgebras, *Adv. Math.*, **139**, (1998) 1–55.
- [13] B. A. Khesin, V. Lyubashenko, and C. Roger, Extensions and contractions of the Lie algebra of q -pseudodifferential symbols on the circle, *J. Funct. Anal.*, **143**, (1997) 55–97.
- [14] O. S. Kravchenko and B. A. Khesin, Central extension of the algebra of pseudodifferential symbols, *Funct. Anal. Appl.*, **25**, (1991) 83–85.
- [15] D. Leites and I. Shchepochkina, preprint MPIM2003–28.
- [16] C. Martinez and E. I. Zelmanov, Simple finite-dimensional Jordan superalgebras of prime characteristic, *J. Algebra*, **236**, (2001) 575–629.
- [17] C. Martinez and E. I. Zelmanov, Lie superalgebras graded by $P(n)$ and $Q(n)$, *Proc. Natl. Acad. Sci. USA*, **100**, (2003) 8130–8137.
- [18] V. Ovsienko and C. Roger, Deforming the Lie algebra of vector fields on S^1 inside the Poisson algebra on \dot{T}^*S^1 . *Commun. Math. Phys.*, **198**, (1998) 97–110.
- [19] V. Ovsienko and C. Roger, Deforming the Lie algebra of vector fields on S^1 inside the Lie algebra of pseudodifferential symbols on S^1 , *Amer. Math. Soc. Transl.*, **194**, (1999) 211–226.
- [20] E. Poletaeva, A spinor-like representation of the contact superconformal algebra $K'(4)$, *J. Math. Phys.*, **42**, (2001) 526–540.
- [21] E. Poletaeva, On the exceptional $N = 6$ superconformal algebra, *J. Math. Phys.*, **46**, (2005) 103504, 13 pp. Publisher’s note, *J. Math. Phys.*, **47**, (2006) 019901, 1 p.
- [22] I. Shchepochkina, hep-th/9702121.
- [23] I. Shchepochkina, Five simple exceptional Lie superalgebras of vector fields, *Funktsional. Anal. i Prilozhen.*, **33**, (1999) 59–72. Translation in *Funct. Anal. Appl.*, **33**, (1999) 208–219.
- [24] I. Shchepochkina, The five exceptional simple Lie superalgebras of vector fields and their fourteen regratings, *Represent. Theory*, **3**, (1999) 373–415.